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ISOMORPHISMS OF THE LATTICE OF INNER IDEALS OF CERTAIN QUADRATIC JORDAN ALGEBRAS

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ABSTRACT. The inner ideals play a role in the theory of quadratic Jordan algebras analogous to that played by the one-sided ideals in the associative theory. In particular, the simple quadratic Jordan algebras satisfying the minimum condition on principal inner ideals play a role analogous to that of the simple artinian algebras in the associative theory. In this paper, we investigate the automorphism group of the lattice of inner ideals of simple quadratic Jordan algebras satisfying the minimum condition on principal inner ideals. For the case $\mathfrak{H}(\mathfrak{A},^*)$ where $(\mathfrak{A},^*)$ is a simple artinian algebra with hermitian involution, we show that the automorphism group of the lattice of inner ideals is isomorphic to the group of semilinear automorphisms of $\mathfrak{A}$. For the case $\mathfrak{H}(\mathfrak{Q}_n,^*)$ where $\mathfrak{Q}$ is a split quaternion algebra, we obtain only a partial result. For the cases $J = \mathfrak{H}(\mathfrak{O}_3)$ and $J = \mathrm{Jord}(Q, 1)$ for $\mathfrak{O}$ an octonion algebra, $(Q,1)$ a nondegenerate quadratic form with base point of Witt index at least three and J finite dimensional, it is shown that every automorphism of the lattice of inner ideals is induced by a norm semisimilarity. Finally, we determine conditions under which two algebras of the type under consideration can have isomorphic lattices of inner ideals.

Preliminaries. We recall some of the basic results concerning inner ideals. First, we recall that a *quadratic Jordan algebra* [12] is a triple $(\mathfrak{J}, U, 1)$ where $\mathfrak{J}$ is a module over a commutative ring Φ with unit, U is a quadratic mapping of $\mathfrak{J}$ into End $\mathfrak{J}$ and 1 is an element of $\mathfrak{J}$ satisfying for all commutative ring extensions of Φ:

(QJ1) $U_1 = 1$;

(QJ2) $U_a U_b U_a = U_{bU_a}$;

(QJ3) if $V_{a,b}$ is defined by $xV_{a,b} = aU_{x,b}$, where $U_{x,y}$ is the bilinearization of U_x, then $U_b V_{a,b} = V_{b,a} U_b$.

One obtains a Jordan algebra $\mathfrak{A}^+$ from an associative algebra $\mathfrak{A}$ by defining $aU_b = bab$. Based on this example, one defines an *inner ideal* $\mathfrak{C}$ in a Jordan algebra $\mathfrak{J}$ to be a Φ-submodule of $\mathfrak{J}$ satisfying $xU_c \in \mathfrak{J}$ for any $x \in \mathfrak{J}$, $c \in \mathfrak{C}$. If b is an element of a Jordan algebra $\mathfrak{J}$, then $\mathfrak{J}U_b$ is an inner ideal in $\mathfrak{J}$, called the *principal inner ideal determined by b*. The element b need not be an element of $\mathfrak{J}U_b$. However, all the algebras we will consider are *regular* (in the sense of von Neumann); i.e. $b \in \mathfrak{J}U_b$ for all $b \in \mathfrak{J}$.

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If $b \neq 0$ is an element of a Jordan algebra $\mathfrak{J}$, then b is said to be a *point* (*element of rank one*) if $\mathfrak{J}U_b = \Phi b$. A subspace $\mathfrak{P}$ of $\mathfrak{J}$ is said to be a *point space* if every element of $\mathfrak{P}$ is a point. It is clear that every subspace of a point space is a point space and that every point space is an inner ideal.

If a is an invertible element of $\mathfrak{J}$, we can form a new Jordan algebra, the *a-isotope* $\mathfrak{J}^{(a)}$, with the same module structure as $\mathfrak{J}$, unit $1^{(a)} = a^{-1}$ and U operator $U_x^{(a)} = U_a U_x$. It is easily seen that a Jordan algebra and its isotopes have the same inner ideals. An isomorphism of a Jordan algebra onto an isotope is called an *isotopy*. The set of all isotopies of a Jordan algebra $\mathfrak{J}$ forms a group called the *structure group* of $\mathfrak{J}$.

In some of our discussion of inner ideals in Jordan algebras, the special universal envelope will play an important role. Thus we presently define it. If $\mathfrak{J}$ is a Jordan algebra, a pair $(\mathfrak{S}, \sigma)$ consisting of an associative algebra $\mathfrak{S}$ with 1 and a homomorphism σ of $\mathfrak{J}$ into $\mathfrak{S}^+$ is called a *special universal envelope* if for any homomorphism φ of $\mathfrak{J}$ into an algebra $\mathfrak{B}^+$, where $\mathfrak{B}$ is associative with 1, there exists a unique homomorphism η of $\mathfrak{S}$ into $\mathfrak{B}$ sending 1 into 1 such that $\sigma\eta = \varphi$.

McCrimmon's second structure theorem [12] states that any simple quadratic Jordan algebra satisfying the minimum condition on principal inner ideals is isotopic to one of the following:

I. A Jordan division algebra.

II. An outer ideal containing 1 in a Jordan algebra Jord$(Q,1)$ where $(Q,1)$ is a regular quadratic form with base point.

III. Δ_n^+ for Δ an associative division ring and $n \geqslant 2$.

IV. An outer ideal containing 1 in an algebra $H(\Delta_n,^*)$ where Δ is an associative division ring, $n \geqslant 2$, with involution * given by $(a_{ij})^* = (\overline{a_{ji}})$ where $a \to \bar{a}$ is an involution in the division ring Δ.

V. $H(\mathfrak{Q}_n,^*)$ for $\mathfrak{Q}$, a split quaternion algebra over its center Φ, with $(a_{ij})^* = (\overline{a_{ji}})$ where $a \to \bar{a}$ is the usual involution in $\mathfrak{Q}$.

VI. $H(\mathfrak{O}_3,^*)$ for $\mathfrak{O}$ an octonion algebra with involution * given by $(a_{ij})^* = (\overline{a_{ji}})$ where $a \to \bar{a}$ is the usual involution in $\mathfrak{O}$.

Over fields of characteristic unequal to two, an outer ideal containing 1 in a Jordan algebra must be the whole algebra. Thus in this case, we get instead of II and IV the following classification due to Jacobson and Osborn [7, Chapter 4]:

II′. A Jordan algebra Jord$(Q,1)$ where Q is a regular quadratic form.

IV′. A Jordan algebra $H(\Delta_n,^*)$ where Δ is an associative division ring, $n \geqslant 2$, with involution * given by $(a_{ij})^* = (\overline{a_{ji}})$ where $a \to \bar{a}$ is an involution in the division ring Δ.

We are now ready to begin our case by case investigation of the automorphism group of the lattice of inner ideals of simple quadratic Jordan algebras satisfying the minimum condition on principal inner ideals.

Since the only inner ideals in a Jordan division algebra $\mathfrak{J}$ are 0 and $\mathfrak{J}$, we always assume our Jordan algebras are not division algebras.

2. The algebras $\mathfrak{A}^+$. In this section, we suppose that $\mathfrak{J} = \mathfrak{A}^+$ where $\mathfrak{A}$ is a simple artinian algebra. We will show that the automorphism group of the lattice of inner ideals of $\mathfrak{A}^+$ is isomorphic to the automorphism group of the ring $\mathfrak{A} \oplus \mathfrak{A}^{op}$ which by a result of Jacobson and Rickart [10] is the special universal envelope of $\mathfrak{J}$.

In this case, McCrimmon [14] has proved that every inner ideal $\mathfrak{C}$ is of the form $e\mathfrak{A}f$ where e and f are idempotents in $\mathfrak{A}$. We note that $e\mathfrak{A}f = e\mathfrak{A} \cap \mathfrak{A}f$; thus every inner ideal in $\mathfrak{A}^+$ is an intersection of a right ideal in $\mathfrak{A}$ and a left ideal in $\mathfrak{A}$. Suppose $\mathfrak{A}f \neq 0$; then $\mathfrak{A}f\mathfrak{A} = \mathfrak{A}$ since $\mathfrak{A}$ is simple. Thus $e\mathfrak{A} = e\mathfrak{A}f\mathfrak{A} = \mathfrak{C}\mathfrak{A}$. Similarly if $e\mathfrak{A} \neq 0$, $\mathfrak{A}f = \mathfrak{A}\mathfrak{C}$. Thus we have proved the following proposition.

Proposition 1. *Suppose $\mathfrak{A}$ is a simple artinian ring, and $\mathfrak{C}$, $\mathfrak{C}'$ are inner ideals in $\mathfrak{A}^+$. Then there exist right ideals $\mathfrak{R}$, $\mathfrak{R}'$ and left ideals $\mathfrak{L}$, $\mathfrak{L}'$ in $\mathfrak{A}$ such that $\mathfrak{C} = \mathfrak{R} \cap \mathfrak{L}$ and $\mathfrak{C}' = \mathfrak{R}' \cap \mathfrak{L}'$. Suppose $\mathfrak{C} \supseteq \mathfrak{C}' \neq 0$; then $\mathfrak{R} \supseteq \mathfrak{R}'$ and $\mathfrak{L} \supseteq \mathfrak{L}'$. Hence if $\mathfrak{C} = \mathfrak{C}' \neq 0$, then $\mathfrak{R} = \mathfrak{R}'$ and $\mathfrak{L} = \mathfrak{L}'$. Furthermore if $\mathfrak{C} = 0$, either $\mathfrak{R} = 0$ or $\mathfrak{L} = 0$.*

We consider $\mathfrak{A}^+$ as $\mathfrak{J} = H(\mathfrak{A} \oplus \mathfrak{A}^{op}, {}^*)$ where $*$ is the exchange involution. Suppose $\mathfrak{C} = e\mathfrak{A}f$ is an inner ideal in $\mathfrak{A}^+$. Then $\mathfrak{C} = \{(eaf, eaf) \mid a \in \mathfrak{A}\}$. The right ideal $\mathfrak{C}(\mathfrak{A} \oplus \mathfrak{A}^{op})$ generated by $\mathfrak{C}$ in $\mathfrak{A} \oplus \mathfrak{A}^{op}$ is equal to $\{(eaf\mathfrak{A}, \mathfrak{A}eaf) \mid a \in \mathfrak{A}\}$ which using the simplicity of $\mathfrak{A}$ is equal to $e\mathfrak{A} \oplus \mathfrak{A}f$ provided $e\mathfrak{A}$ and $\mathfrak{A}f$ are unequal to zero. Conversely, $(e\mathfrak{A} \oplus \mathfrak{A}f) \cap \mathfrak{J} = (e\mathfrak{A}f \oplus e\mathfrak{A}f) \cap \mathfrak{J} = \mathfrak{C}$. This yields an inclusion-preserving injection of the nonzero inner ideals in $\mathfrak{J}$ to the right ideals of $\mathfrak{E}(\mathfrak{A}) = \mathfrak{A} \oplus \mathfrak{A}^{op}$.

Since $\mathfrak{E}(\mathfrak{A})$ is semisimple artinian, any right ideal has the form $a\mathfrak{E}$ for some idempotent a in $\mathfrak{E} = \mathfrak{E}(\mathfrak{A})$. But then $a = e + f$, $e \in \mathfrak{A}$, $f \in \mathfrak{A}^{op}$, will imply that e and f are idempotents and that $a\mathfrak{E} = e\mathfrak{A} \oplus \mathfrak{A}f$.

Thus we have proved the following lemma.

Lemma 1. *The mapping $\rho : \mathfrak{C} \to \mathfrak{C}\mathfrak{E}$ is an inclusion-preserving bijection of the set of nonzero inner ideals of $\mathfrak{A}^+$ and the set $\mathscr{S}$ of all right ideals of $e\mathfrak{A} \oplus \mathfrak{A}f$ in $\mathfrak{E} = \mathfrak{A} \oplus \mathfrak{A}^{op}$ with neither e nor f equal to zero. Furthermore,*

$$\rho^{-1} : a\mathfrak{E} \to a\mathfrak{E} \cap \mathfrak{J}.$$

Now suppose $\pi \in \operatorname{Aut} \mathscr{I}$ ($\mathscr{I}$ the lattice of inner ideals of $\mathfrak{J}$). Then $\psi = \rho^{-1}\pi\rho$ is an inclusion-preserving bijection of the subset $\mathscr{S}$ of the lattice $\mathscr{R}(\mathfrak{E})$ of right ideals of $\mathfrak{E}$. Thus it is clear that the automorphism group of the lattice of inner ideals of $\mathfrak{A}^+$ is isomorphic to the group of inclusion-preserving bijections of $\mathscr{S}$.

Let $\mathscr{R}$ denote the sublattice of $\mathscr{R}(\mathfrak{E})$ consisting of all right ideals of the form

$e\mathfrak{A} \oplus \mathfrak{A}$ and $\mathscr{L}$ denote the sublattice of $\mathscr{R}(\mathfrak{E})$ consisting of all right ideals of the form $\mathfrak{A} \oplus \mathfrak{A}f$.

Suppose ψ is an inclusion-preserving bijection of $\mathscr{S}$ or an automorphism of $\mathscr{R}(\mathfrak{E})$. Then we would like to show that $(\mathscr{L} \cup \mathscr{R})\psi \subseteq \mathscr{L} \cup \mathscr{R}$.

Suppose $\mathfrak{L}\psi \nsubseteq \mathscr{L} \cup \mathscr{R}$. Then since $\mathfrak{E}$ is aritinian, hence noetherian, choose $\mathfrak{A} \oplus \mathfrak{A}f$ maximal with $(\mathfrak{A} \oplus \mathfrak{A}f) \notin \mathscr{L} \cup \mathscr{R}$. $\mathfrak{A}$, being a simple artinian ring, is isomorphic to the ring of endomorphisms of an n-dimensional vector space V over a division ring Δ. From now on, we assume $n \geqslant 3$. Then $\mathscr{L}$, being isomorphic to the lattice of the ideals of $\mathfrak{A}$, is isomorphic to the lattice of subspaces of V. Suppose $\mathfrak{A} \oplus \mathfrak{A}f$ corresponds to W in this isomorphism. If the codimension of W is at least two, there exist subspaces W_1, W_2, and W_3 properly containing W with $W_i \cap W_j = W$ if $i \neq j$. If $\mathfrak{A} \oplus \mathfrak{B}_i$ corresponds to W_i, then two of $(\mathfrak{A} \oplus \mathfrak{B}_i)\psi$ lie in $\mathscr{L}$ or two of them lie in $\mathscr{R}$ (by the maximality of $\mathfrak{A} \oplus \mathfrak{A}f$). Thus $(\mathfrak{A} \oplus \mathfrak{A}f)\psi = (\mathfrak{A} \oplus \mathfrak{B}_i)\psi \cap (\mathfrak{A} \oplus \mathfrak{B}_j)\psi$ lies in $\mathscr{L}$ or $\mathscr{R}$, a contradiction. If the codimension of W is equal to one, then $\mathfrak{A} \oplus \mathfrak{A}f$ is a maximal right ideal in E, but its image cannot be (since the maximal right ideals of E are all in $\mathscr{L} \cup \mathscr{R}$), another contradiction.

Thus $\mathscr{L}\psi \subseteq \mathscr{L} \cup \mathscr{R}$ and similarly $\mathscr{R}\psi \subseteq \mathscr{L} \cup \mathscr{R}$. Now suppose there exist $\mathfrak{R}$, $\mathfrak{L} \neq 0$, $\mathfrak{A}$ respectively a right and a left ideal in $\mathfrak{A}$ such that $(\mathfrak{A} \oplus \mathfrak{A}f_1)\psi = \mathfrak{A} \oplus \mathfrak{L}$ and $(\mathfrak{A} \oplus \mathfrak{A}f_2)\psi = \mathfrak{R} \oplus \mathfrak{A}$ for some idempotents f_1 and f_2 in $\mathfrak{A}$. Then

$$(\mathfrak{A} \oplus (\mathfrak{A}f_1 \cap \mathfrak{A}f_2))\psi = \mathfrak{R} \oplus \mathfrak{L},$$

a contradiction. Thus either $\mathscr{L}\psi \subseteq \mathscr{L}$ or $\mathscr{L}\psi \subseteq \mathscr{R}$. Then working also with ψ^{-1}, we obtain either $\mathscr{L}\psi = \mathscr{L}$ and $\mathscr{R}\psi = \mathscr{R}$ or $\mathscr{L}\psi = \mathscr{R}$ and $\mathscr{R}\psi = \mathscr{L}$.

Suppose $\mathscr{L}\psi = \mathscr{L}$. Then by the fundamental theorem of projective geometry, there exist ring automorphisms η, ζ of $\mathfrak{A}$ such that $(\mathfrak{A} \oplus \mathfrak{L})\psi = \mathfrak{A} \oplus \mathfrak{L}\eta$ and $(\mathfrak{R} \oplus \mathfrak{A})\psi = \mathfrak{R}\zeta \oplus \mathfrak{A}$ for all left ideals $\mathfrak{L}$ and right ideals $\mathfrak{R}$ of $\mathfrak{A}$. Then $(\mathfrak{R} \oplus \mathfrak{L})\psi = \mathfrak{R}\zeta \oplus \mathfrak{L}\eta$ is an automorphism of $\mathscr{R}(\mathfrak{E})$.

Now suppose $\mathscr{L}\psi = \mathscr{R}$. Then the lattice of left ideals of $\mathfrak{A}$ is isomorphic to the lattice of right ideals of $\mathfrak{A}$. Thus, by the fundamental theorem of projective geometry, it follows that $\mathfrak{A}$ has an anti-automorphism φ. Then we regard φ as an automorphism of $\mathfrak{E}$ which switches the components. Then $(\mathfrak{A} \oplus \mathfrak{L})\psi\varphi = \mathfrak{A} \oplus \mathfrak{L}'$ and $(\mathfrak{R} \oplus \mathfrak{A})\psi\varphi = \mathfrak{R}' \oplus \mathfrak{A}$ for some left, right ideals $\mathfrak{L}'$, $\mathfrak{R}'$ of $\mathfrak{A}$. Thus there exist ring automorphisms η and ζ of $\mathfrak{A}$ such that $(\mathfrak{A} \oplus \mathfrak{L})\psi\varphi = \mathfrak{A} \oplus \mathfrak{L}\eta$ and $(\mathfrak{R} \oplus \mathfrak{A})\psi\varphi = \mathfrak{R}\zeta \oplus \mathfrak{A}$ for all left, right ideals $\mathfrak{L}$, $\mathfrak{R}$ of $\mathfrak{A}$. Then $(\mathfrak{R} \oplus \mathfrak{L})\psi = (\mathfrak{R}\zeta \oplus \mathfrak{L}\eta)\varphi^{-1}$ is an automorphism of $\mathscr{R}(\mathfrak{E})$. Thus we have proved the following proposition.

Proposition 2. *Suppose $\mathfrak{A} = \Delta_n$ is a simple artinian ring with $n \geqslant 3$. Then any automorphism of $\mathscr{R}(\mathfrak{E})$ stabilizes $\mathscr{S} = \{e\mathfrak{A} \oplus \mathfrak{A}f \mid e\mathfrak{A}, \mathfrak{A}f \neq 0\}$, and any inclusion-preserving bijection of $\mathscr{S}$ can be extended uniquely to an automorphism of $\mathscr{R}(\mathfrak{E})$.*

Proof. Everything is proved in the discussion before the statement of the proposition except for the uniqueness which follows since $\mathscr{S}$ generates $\mathscr{R}(\mathfrak{E})$.

Lemma 2. *Suppose $\mathfrak{A} = \Delta_n$ is a simple artinian ring with $n \geqslant 3$. Let η be an automorphism of the ring $\mathfrak{A}$ such that $\mathfrak{R}\eta = \mathfrak{R}$ for all right ideals $\mathfrak{R}$ of $\mathfrak{A}$. Then η is the identity transformation.*

Proof. Suppose $\mathfrak{A} = \mathrm{End}_\Delta V$. Then the lattice of right ideals of $\mathfrak{A}$ is anti-isomorphic to the lattice of subspaces of V. Thus it suffices to show that the automorphism group of the lattice of subspaces of V is the automorphism group of the ring $\mathfrak{A}$.

Every automorphism η of $\mathfrak{A}$ has the form $X \to S^{-1}XS$ where S is a semilinear automorphism of V. Furthermore, η is the identity if and only if S is a scalar. Thus the automorphism group of the ring $\mathfrak{A}$ is equal to the quotient group of the group of semilinear automorphisms of V by the group of scalars. By the fundamental theorem of projective geometry, this latter group is equal to the automorphism group of the lattice of subspaces of V.

Proposition 3. Aut $\mathscr{R}(\mathfrak{E})$ *is isomorphic to the automorphism group of the ring* $\mathfrak{E}$.

Proof. In the proof of Proposition 2, we proved that every automorphism of $\mathscr{R}(\mathfrak{E})$ is induced by an automorphism of $\mathfrak{E}$. Thus suppose $\psi \in \mathrm{Aut}\,\mathfrak{E}$ induces the identity on $\mathscr{R}(\mathfrak{E})$. Then $\psi = (\zeta, \eta)$ since ψ cannot interchange $\mathscr{L}$ and $\mathscr{R}$. Then by Lemma 2, ζ and η must be the identity mappings.

Thus we have proved the following theorem.

Theorem 1. *Suppose $\mathfrak{A} = \Delta_n$ is a simple artinian ring with $n \geqslant 3$. Then the automorphism group of the lattice of inner ideals of $\mathfrak{A}^+$ is isomorphic to the automorphism group of the ring $\mathfrak{A} \oplus \mathfrak{A}^{\mathrm{op}}$, the special universal envelope of the Jordan algebra $\mathfrak{A}^+$.*

We would now like to show that if two simple artinian rings $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the hypotheses of Theorem 1 and are such that $\mathfrak{A}^+$ and $\mathfrak{B}^+$ have isomorphic lattices of inner ideals, then $\mathfrak{A}^+$ and $\mathfrak{B}^+$ are isomorphic.

First, we need the following lemma whose proof depends on the fact that any left ideal in $\mathrm{End}_\Delta V$ is the set of all linear transformations of V whose images lie in a fixed subspace of V.

Lemma 3. *If $\mathfrak{A}$ is a simple artinian ring, then any left ideal in $\mathfrak{A}$ is an intersection of maximal left ideals of $\mathfrak{A}$.*

Now let η be an isomorphism of the lattice of inner ideals of $\mathfrak{A}^+$ onto the lattice of inner ideals of $\mathfrak{B}^+$. Then we note that the maximal inner ideals are precisely the maximal one-sided ideals. Thus any maximal one-sided ideal of $\mathfrak{A}$ is mapped to a maximal one-sided ideal of $\mathfrak{B}$.

Suppose $\mathfrak{A}e_1$ and $\mathfrak{A}e_2$ are two maximal left ideals such that $(\mathfrak{A}e_1)\eta = \mathfrak{B}f_1$ and $(\mathfrak{A}e_2)\eta = f_2\mathfrak{B}$. Then $\mathfrak{A}e_1 \cap \mathfrak{A}e_2 \neq 0$ since $\mathfrak{B}f_1 \cap f_2\mathfrak{B} \neq 0$ (Proposition 1). It is clear that we can choose a maximal left ideal $\mathfrak{A}e_3$ such that $\mathfrak{A}e_2 \neq \mathfrak{A}e_3$ and $\mathfrak{A}e_1 \cap \mathfrak{A}e_2 = \mathfrak{A}e_1 \cap \mathfrak{A}e_3$. Let $\mathfrak{A}e = \mathfrak{A}e_1 \cap \mathfrak{A}e_2$. Then

$$(\mathfrak{A}e)\eta = (\mathfrak{A}e_1)\eta \cap (\mathfrak{A}e_2)\eta = \mathfrak{B}f_1 \cap f_2\mathfrak{B}.$$

But $(\mathfrak{A}e)\eta = (\mathfrak{A}e_1)\eta \cap (\mathfrak{A}e_3)\eta = \mathfrak{B}f_1 \cap (\mathfrak{A}e_3)\eta$. If $(\mathfrak{A}e_3)\eta$ is a maximal left ideal, then $\mathfrak{B}f_1 \cap f_2\mathfrak{B}$ is a left ideal, a contradiction. If $(\mathfrak{A}e_3)\eta$ is a right ideal, then by Proposition 1, $(\mathfrak{A}e_3)\eta = f_2\mathfrak{B}$ contradicting the injectivity of η. Thus either all maximal left ideals are mapped to maximal left ideals, or they are all mapped to maximal right ideals.

If every maximal left ideal is mapped to a maximal left ideal, then by Lemma 3 the lattice of left ideals of $\mathfrak{A}$ is isomorphic to the lattice of left ideals of $\mathfrak{B}$. Hence, by the fundamental theorem of projective geometry, $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic rings. If every maximal left ideal is mapped to a maximal right ideal, then the lattice of left ideals of $\mathfrak{A}$ is isomorphic to the lattice of right ideals of $\mathfrak{B}$. Thus $\mathfrak{A}$ and $\mathfrak{B}^{op}$ are isomorphic rings. Hence we have almost proved the following theorem.

Theorem 2. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are simple artinian rings one of which contains a set of three orthogonal idempotents. If the lattices of inner ideals of $\mathfrak{A}^+$ and $\mathfrak{B}^+$ are isomorphic, there exists a semilinear algebra isomorphism of $\mathfrak{A}^+$ onto $\mathfrak{B}^+$. Hence the lattices of inner ideals of $\mathfrak{A}^+$ and $\mathfrak{B}^+$ are isomorphic if and only if their special universal envelopes are isomorphic rings.*

Proof. With no loss of generality, suppose $\mathfrak{A} = \Delta_n$ with Δ a division ring and $n \geqslant 3$. Then by an argument on the length of a maximal chain of inner ideals [14, p. 458], it follows that $\mathfrak{B} = \Delta'_n$ for some division ring Δ'. Then the argument given before the statement of the theorem completes the proof.

3. **The algebras $\mathfrak{H}(\Delta_n,{}^*)$.** In this section, we will consider Jordan algebras of the form $\mathfrak{H}(\Delta_n,{}^*)$ where $n \geqslant 3$ and * is a hermitian involution. We will prove that the automorphism group of the lattice of inner ideals of $\mathfrak{J} = \mathfrak{H}(\Delta_n,{}^*)$ is isomorphic to the group of ring automorphisms of Δ_n, which by a result due to Jacobson and Rickart [11] is the special universal envelope of $\mathfrak{J}$.

McCrimmon [14] has proved that every inner ideal in $\mathfrak{J}$ is of the form $b\mathfrak{J}b$ for some b in $\mathfrak{J}$; alternatively, they are of the form $e^*\mathfrak{J}e$ for some idempotent e in Δ_n.

Proposition 4. *Suppose $\mathfrak{J} = \mathfrak{H}(\mathfrak{A},{}^*)$ where $\mathfrak{A} = \Delta_n$, Δ is a division ring, $n \geqslant 3$ and * is a hermitian involution. Then the mappings*

$\zeta : \mathfrak{C} \to \mathfrak{C}\mathfrak{A}$, *$\mathfrak{C}$ an inner ideal in $\mathfrak{J}$,*

$\eta : \mathfrak{R} \to \mathfrak{R} \cap \mathfrak{J}$, *$\mathfrak{R}$ a right ideal in $\mathfrak{A}$,*

establish an isomorphism between the lattice of inner ideals of $\mathfrak{J}$ and the lattice of right ideals of $\mathfrak{A}$.

In order to prove Proposition 4, we first need the following lemma.

Lemma 4. *Suppose $e\mathfrak{A}$ is a right ideal in $\mathfrak{A} = \Delta_n$. Then there exists b in $\mathfrak{J} = \mathfrak{H}(\mathfrak{A},{}^*)$ such that $e\mathfrak{A} = b\mathfrak{A}$.*

Proof. There exists an integer $i > 0$ and an invertible element u in $\mathfrak{A}$ such that $ue\mathfrak{A}$ is the set of all matrices with the first i rows equal to zero. Thus $ue\mathfrak{A} = b_i\mathfrak{A}$ where b_i is the diagonal matrix with the first i rows equal to zero and 1's on the diagonal beginning with row $i + 1$. Hence $e\mathfrak{A} = u^{-1}b_i\mathfrak{A} = u^{-1}b_i(u^{-1})^*\mathfrak{A}$. Let $b = u^{-1}b_i(u^{-1})^*$. Then $e\mathfrak{A} = b\mathfrak{A}$ and b is in $\mathfrak{J}$.

Proof of Proposition 4. Choose c in $\mathfrak{J}$ such that $\mathfrak{R} = c\mathfrak{A}$. Since $\mathfrak{A}$ is simple, there exists an idempotent e in $\mathfrak{A}$ with $\mathfrak{R} = e\mathfrak{A}$.

$$\mathfrak{R}\eta = (c\mathfrak{A})\eta = c\mathfrak{A} \cap \mathfrak{J} = c\mathfrak{A} \cap \mathfrak{A}c \cap \mathfrak{J} = e\mathfrak{A}e^* \cap \mathfrak{J} = c\mathfrak{A}c \cap \mathfrak{J} = c\mathfrak{J}c$$

[14, Proposition 3].

Choose b in $\mathfrak{J}$ such that $\mathfrak{C} = b\mathfrak{J}b$. Then $\mathfrak{C}\zeta = (b\mathfrak{J}b)\zeta = (b\mathfrak{J}b)\mathfrak{A}$. It is a simple computation using the regularity of $\mathfrak{J}$ to show that $(b\mathfrak{J}b)\mathfrak{A} = b\mathfrak{A}$.

Thus $\eta\zeta$ and $\zeta\eta$ are the identity mappings, as required.

Theorem 3. *Suppose $\mathfrak{J} = \mathfrak{H}(\Delta_n,^*)$ where Δ is a division ring, $n \geqslant 3$ and * is a hermitian involution. Then the automorphism group of the lattice of inner ideals of $\mathfrak{J}$ is isomorphic to the automorphism group of the ring Δ_n, the special universal envelope of $\mathfrak{J}$.*

Proof. By Proposition 4, it suffices to show that the automorphism group of the lattice of right ideals of Δ_n is the group of ring automorphisms of Δ_n. By the fundamental theorem of projective geometry, it follows that every automorphism of the lattice of right ideals of Δ_n is induced by a ring automorphism of Δ_n. The theorem then follows from Lemma 2.

We now want to consider under what conditions two Jordan algebras of the type considered in this section can have isomorphic lattices of inner ideals. The result is the following theorem.

Theorem 4. *Suppose $\mathfrak{J} = \mathfrak{H}(\Delta_n,^*)$ and $\mathfrak{J}' = \mathfrak{H}(\Delta'_m,^{*'})$ are such that Δ, Δ' are division rings, n or $m \geq 3$, and *, $^{*'}$ are hermitian involutions. Then the lattices of inner ideals of $\mathfrak{J}$ and $\mathfrak{J}'$ are isomorphic if and only if Δ_n and Δ'_m are isomorphic rings.*

Proof. If the lattice of inner ideals of $\mathfrak{J}$ is isomorphic to the lattice of inner ideals of $\mathfrak{J}'$, then the lattices of right ideals of Δ_n and Δ'_m are isomorphic (Proposition 4) which then implies that Δ_n and Δ'_m are isomorphic. The converse is clear.

We now present an example of two Jordan algebras with isomorphic lattices of inner ideals but which are not isotopic. Let n be an integer at least 3 and let $\mathbf{C}$ be the field of complex numbers. If $x \in \mathbf{C}_n$, define x^* to be the transpose of x and $x^{*'}$ to be the conjugate transpose of x. Then $\mathfrak{H}(\mathbf{C}_n,^*)$ and $\mathfrak{H}(\mathbf{C}_n,^{*'})$ are Jordan algebras over the field of real numbers. It is clear that they have isomorphic lattices of inner ideals, but, since they are not of the same dimension, they cannot be isotopic.

4. The algebra $\mathfrak{H}(\mathfrak{O}_n,{}^*)$. In this section, we assume $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_n,{}^*)$ where $\mathfrak{O} = \Phi_2$ is a split quaternion algebra over its center Φ and $n \geqslant 3$. We can also consider $\mathfrak{J}$ as the algebra of *selfadjoint transformations relative to an alternating bilinear form* in a $2n$-dimensional vector space over the field Φ. In this case, we will show that the automorphism group of the lattice of inner ideals is a subgroup of the group of ring automorphisms of $\mathfrak{O}_n$, the special universal envelope of $\mathfrak{J}$. It remains an open question whether or not the two groups are equal.

McCrimmon [14] has proved that in this case the inner ideals are either of the form $e^*\mathfrak{J}e$ where e is an idempotent in $\mathfrak{O}_n$ or they are point spaces. Furthermore, every maximal point space has the form $\mathfrak{P} = \Phi e_1 + \mathfrak{O}\varepsilon\,[12]$ or $\mathfrak{P} = \Phi e_1 + \varepsilon\mathfrak{O}\,[12] + \dots + \varepsilon\mathfrak{O}[1n]$ in a suitable isotope where ε is a primitive idempotent in $\mathfrak{O}$, $a[ij] = a\gamma_j e_{ij} + \bar{a}\gamma_i e_{ji}$ relative to a set of matrix units e_{ij} where $\gamma_k \neq 0 \in \Phi$.

The set of inner ideals of the form $e\mathfrak{J}e^*$ has many of the properties of the lattice of inner ideals in the previous two cases.

It is clear that the inner ideals of the form $e\mathfrak{J}e^*$ where e is a transformation of rank $2n - 1$ are maximal inner ideals. We now show that the maximal point spaces of dimension $2n - 1$ are also maximal point spaces.

Lemma 5. *The maximal point spaces of dimension* $2n - 1$ *in* $\mathfrak{H}(\mathfrak{O}_n,{}^*)$ *are maximal inner ideals.*

Proof. Suppose the lemma is false; then there exists a point space $\mathfrak{P}$ of dimension $2n - 1$ which can be imbedded in an inner ideal of the form $e\mathfrak{J}e^*$. Then by passing to an isotope, we can assume $\mathfrak{P} = \Phi e_1 + \varepsilon\mathfrak{O}[12] + \dots + \varepsilon\mathfrak{O}[1n]$ where $\varepsilon = e_{11}$ (in the identification of $\mathfrak{O}$ with Φ_2). If $\mathfrak{P}$ is contained in $e\mathfrak{J}e^*$, then $\mathfrak{P}$ is contained in $e\mathfrak{O}_n$, and we shall show this to be impossible.

We can suppose the rank of e is $2n - 1$; then there exists an invertible v in $\mathfrak{O}_n$ such that $ve\mathfrak{O}_n$ is the set of all matrices in Φ_{2n} with last row equal to zero.

Let $v = \sum_{i,j=1}^{2n} a_{ij}e_{ij}, a_{ij} \in \Phi$. The arbitrary element c of $\mathfrak{P}$ has the form $c = \alpha e_{11} + b_{13}e_{13} + b_{14}e_{14} + \dots + b_{1,2n}e_{1,2n} + \alpha e_{22} - \gamma_2 b_{14}e_{32} + \gamma_2 b_{13}e_{42} - \dots - \gamma_n b_{1,2n}e_{2n-1,2} + \gamma_n b_{1,2n-1}e_{2n,2}$ where α, b_{ij} are arbitrary elements of Φ and $\gamma_i \neq 0 \in \Phi$.

We require $v\mathfrak{P}$ to have last row equal to zero. Thus we are considering $\sum_j a_{2n,j}c_{j,k}$ where $c = (c_{jk})$.

Let $k = 3$; then $c_{13} = b_{13}$ and $c_{i3} = 0$ if $i \neq 1$. Thus $a_{2n,1}c_{13} = 0$ implies $a_{2n,1} = 0$.

Let $k = 2$; then $c_{22} = \alpha$, $c_{2m,2} = \gamma_m b_{1,2m-1}$ and $c_{2m-1,2} = -\gamma_m b_{1,2m}$ if $m > 1$. Then we have

$$0 = \alpha a_{2n,2} + \sum_{m=2}^{n} a_{2n,2m}c_{2m,2} + \sum_{m=2}^{n} a_{2n,2m-1}c_{2n-1,2}.$$

This gives us

$$0 = \alpha a_{2n,2} + \sum_{m=2}^{n} \gamma_m a_{2n,2m}b_{1,2m-1} - \sum_{m=2}^{n} \gamma_m a_{2n,2m-1}b_{1,2m}.$$

But since α and the b_{ij} are arbitrary and $\gamma_m \neq 0$, we deduce that $a_{2n,m} = 0$ if $m \geq 2$ by taking in turn one of α and the b_{ij} equal to one and all the others zero. Thus the last row of v must be zero, contradicting the assumption that v is invertible.

We now begin our investigation of the automorphisms of the lattice of inner ideals of $\mathfrak{J}$.

Lemma 6. *Any automorphism of the lattice of inner ideals of* $\mathfrak{J} = \mathfrak{H}(\mathfrak{Q}_n,^*)$ *must stabilize* $\{e\mathfrak{J}e^* \mid e^2 = e \in \mathfrak{Q}_n\}$.

Proof. Suppose $\mathfrak{C}$ is an inner ideal of the form $e\mathfrak{J}e^*$. Then if e is a rank one transformation, $\mathfrak{C} = 0$. Thus we assume e is of rank at least two. Then there exists some isotope of $\mathfrak{J}$ in which $\mathfrak{C}$ has either the form

$$(e_{11} + \ldots + e_{mm})\mathfrak{J}(e_{11} + \ldots + e_{mm})$$

or the form

$$(e_{11} + \ldots + e_{mm} + \bar{\varepsilon}e_{nn})\mathfrak{J}(e_{11} + \ldots + e_{mm} + \varepsilon e_{nn})$$

for some $1 \leqslant m \leqslant n - 1$ and ε a primitive idempotent in $\mathfrak{Q}$ [14, Main Theorem].

Now if $\mathfrak{C}$ has either of these two forms, there exist two point spaces $\mathfrak{P}_1 = \Phi e_1 + \varepsilon\mathfrak{Q}[12] + \ldots + \varepsilon\mathfrak{Q}[1m]$ and $\mathfrak{P}_2 = \Phi e_1 + \bar{\varepsilon}\mathfrak{Q}[12] + \ldots + \bar{\varepsilon}\mathfrak{Q}[1m]$ contained in $\mathfrak{C}$ with $\mathfrak{P}_1 \cap \mathfrak{P}_2 = \Phi e_1$, a one-dimensional inner ideal and $\dim \mathfrak{P}_1 = \dim \mathfrak{P}_2 = 2m - 1$.

If $\mathfrak{C}$ is of the first (second) type, then it follows by an argument similar to that used in proving Lemma 5 that the length of a maximal chain of inner ideals from $\mathfrak{C}$ to zero (not including zero) is $2m$ (resp. $2m + 1$).

If $\mathfrak{C}$ is mapped to a point space $\mathfrak{P}$, $\dim \mathfrak{P} = 2m$ or $2m + 1$ in the two cases. If $m \geqslant 3$, it is clearly impossible to have two subspaces each of dimension $2m - 1$ intersecting in a one-dimensional subspace inside a space of dimension $2m + 1$. If $m = 2$ and $\mathfrak{C}$ is of the first type, we obtain the contradiction as for $m \geqslant 3$. If $m = 2$ and $\mathfrak{C}$ is of the second type, we take $\mathfrak{P}_1 = \Phi e_1 + \varepsilon\mathfrak{Q}[12] + \Phi\varepsilon[1n]$ and $\mathfrak{P}_2$ as above and obtain a similar contradiction. It suffices to assume $m \geqslant 2$ since the inner ideals with $m = 1$ are minimal, and all minimal inner ideals are of the form $e\mathfrak{J}e^*$ since they are principal [8, p. 3.11].

Lemma 7. *Let* $\mathfrak{A} = \mathfrak{Q}_n$ *where* $\mathfrak{Q}$ *is a split quaternion algebra and* $n \geq 3$ *and let* $\mathfrak{J} = \mathfrak{H}(\mathfrak{A},^*)$. *Then the mapping* $\mathfrak{C} \rightarrow \mathfrak{C}\mathfrak{A}$ *is an inclusion-preserving bijection of the set of all inner ideals of the form* $e\mathfrak{J}e^*$ *and the set of right ideals of* $\mathfrak{A}$ *generated by idempotents* e *of rank at least two.*

Proof. Suppose e is an idempotent of even rank $2k$ (in Φ_{2n}). Then there exist invertible matrices u and v in $\mathfrak{A}$ such that $uev = \mathrm{diag}\{0, \ldots, 0, 1, \ldots, 1\}$ where the number of ones is $2k$. Then $uev \in \mathfrak{J}$. Let $b = u^{-1}(uev)(u^{-1})^*$. Then $b \in \mathfrak{J}$

and $b\mathfrak{A} = e\mathfrak{A}$ since v and $(u^{-1})^*$ are invertible. Thus given any idempotent e in $\mathfrak{A}$ of even rank, there exists b in $\mathfrak{J}$ such that $e\mathfrak{A} = b\mathfrak{A}$.

Now suppose $\mathfrak{C} = e\mathfrak{J}e^*$. Then $\mathfrak{C} = 0$ if and only if the rank of e is at most one. Thus suppose the rank of e is at least two. If e is of even rank, there exists c in $\mathfrak{J}$ such that $e\mathfrak{A} = c\mathfrak{A}$. Then as in the proof of Proposition 4, $e\mathfrak{J}e^* = e\mathfrak{A} \cap \mathfrak{J} = c\mathfrak{A} \cap \mathfrak{J} = c\mathfrak{J}c$. Then $c\mathfrak{A} = \mathfrak{C}\mathfrak{A}$. Thus $\mathfrak{C}\mathfrak{A} = e\mathfrak{A}$.

Now suppose e is of odd rank, say $2k + 1$. Then $(e\mathfrak{J}e^*)\mathfrak{A} = \mathfrak{C}\mathfrak{A} \subseteq e\mathfrak{A}$. Since e is of rank $2k + 1$, there exists a set $\{e_1, \ldots, e_{2k+1}\}$ of primitive orthogonal idempotents such that $e = e_1 + \ldots + e_{2k+1}$. Let $e' = e_1 + \ldots + e_{2k}$. Then $e'\mathfrak{J}(e')^*\mathfrak{A} = e'\mathfrak{A}$. Thus $e'\mathfrak{A} \subseteq \mathfrak{C}\mathfrak{A}$.

Let $e'' = e_2 + \ldots + e_{2k+1}$. Then since e'' is of even rank $e''\mathfrak{J}(e'')^* = e''\mathfrak{A}$. Thus $e''\mathfrak{A} \subseteq \mathfrak{C}\mathfrak{A}$.

Then we have $e'\mathfrak{A} + e''\mathfrak{A} \subseteq \mathfrak{C}\mathfrak{A} \subseteq e\mathfrak{A}$. But $e \in e'\mathfrak{A} + e''\mathfrak{A}$. Thus $e\mathfrak{A} \subseteq e'\mathfrak{A} + e''\mathfrak{A}$. Hence $e\mathfrak{A} \subseteq \mathfrak{C}\mathfrak{A} \subseteq e\mathfrak{A}$. Thus $\mathfrak{C}\mathfrak{A} = e\mathfrak{A}$, proving the lemma.

Now we are ready to investigate the group of inclusion-preserving bijections of $\{e\mathfrak{J}e^* \mid e^2 = e \in \mathfrak{Q}_n\}$.

Proposition 5. *Suppose* $\mathfrak{J} = \mathfrak{H}(\mathfrak{Q}_n, ^*)$ *where* $\mathfrak{Q}$ *is a split quaternion algebra and* $n \geq 3$. *Then the group of inclusion-preserving bijections of* $\{e\mathfrak{J}e^* \mid e^2 = e \in \mathfrak{Q}_n\}$ *is the automorphism group of the ring* $\mathfrak{Q}_n$.

Proof. By Lemma 7, there exists an inclusion-preserving bijection of $\{e\mathfrak{J}e^* \mid e^2 = e \in \mathfrak{Q}_n\}$ and $\{e\mathfrak{Q}_n \mid e^2 = e \in \mathfrak{Q}_n; \text{rank } e \geqslant 2\}$. It is an immediate consequence of the fundamental theorem of projective geometry [1, p. 88] that any inclusion-preserving bijection of $\{e\mathfrak{Q}_n \mid e^2 = e \in \mathfrak{Q}_n; \text{rank } e \geqslant 2\}$ can be extended uniquely to an automorphism of the lattice of right ideals of $\mathfrak{Q}_n$. But since $\mathfrak{Q}_n$ is simple artinian, the automorphism group of the lattice of right ideals of $\mathfrak{Q}_n$ is isomorphic to the automorphism group of the ring $\mathfrak{Q}_n$ (Lemma 2).

We now want to investigate the automorphism group of the lattice of inner ideals of $\mathfrak{J}$ using the result of Proposition 5. The key result is the following lemma.

Lemma 8. *Let* φ *be an automorphism of the lattice of inner ideals of* $\mathfrak{J} = \mathfrak{H}(\mathfrak{Q}_n, ^*)$ *such that* $(e\mathfrak{J}e^*)\varphi = e\mathfrak{J}e^*$ *for all idempotents* e *in* $\mathfrak{Q}_n$. *Then* φ *is the identity automorphism.*

Proof. Let $\mathfrak{P}$ be a point space with basis $u_1, \ldots, u_r$. Then $\Phi u_i = \mathfrak{J}U_{u_i}$ for each i. Then $(u_i\mathfrak{J}u_i)\mathfrak{A} = u_i\mathfrak{A}$. Then there exists an idempotent e_i in $\mathfrak{Q}_n$ such that $e_i\mathfrak{A} = u_i\mathfrak{A}$. But then $e_i\mathfrak{A} \cap \mathfrak{J} = u_i\mathfrak{A} \cap \mathfrak{J} = u_i\mathfrak{J}u_i$. But $e_i\mathfrak{A} \cap \mathfrak{J} = e_i\mathfrak{J}e_i^*$. Thus $\Phi u_i = e_i\mathfrak{J}e_i^*$. Thus we have $(\Phi u_i)\varphi = \Phi u_i$ for each i. Thus if φ is an automorphism of the lattice of inner ideals, $\mathfrak{P}\varphi = \mathfrak{P}$.

Then it follows from Proposition 5 and Lemmas 6 and 8 that the automorphism group of the lattice of inner ideals of $\mathfrak{J}$ is a subgroup of the automorphism group of the ring $\mathfrak{Q}_n$. It is still an open question whether the two groups are equal. If they are, it follows from Theorems 1 and 3 that, if $\mathfrak{J}$ is a simple special quadratic Jordan algebra of capacity at least three, then the automorphism group of the lattice of inner ideals of $\mathfrak{J}$ is isomorphic to the group of semilinear algebra automorphisms of the special universal envelope of $\mathfrak{J}$.

Lemma 9. *The length of a maximal chain of inner ideals from a maximal inner ideal in* $\mathfrak{J} = \mathfrak{H}(\mathfrak{Q}_n, ^*)$ *to zero (excluding zero) is* $2n - 1$.

Proof. A maximal inner ideal in $\mathfrak{J}$ is either a $(2n-1)$-dimensional point space or is of the form $e\mathfrak{J}e^*$ for e an idempotent of rank $2n-1$ by the Main Theorem of [14] and Lemma 5. Since every subspace of a point space is a point space, the result is clear for $(2n-1)$-dimensional point spaces. A maximal inner subideal of a maximal $e\mathfrak{J}e^*$ is either a $(2n-2)$-dimensional point space or a space $f\mathfrak{J}f^*$ where f is an idempotent of rank $2n-2$. Such an inner ideal is isotopic to $\mathfrak{H}(\mathfrak{Q}_{n-1}, ^*)$ [14, pp. 450-457], and the result then follows by an easy induction.

We will now use Lemmas 6 and 7 to deduce necessary and sufficient conditions for the lattice of inner ideals of $\mathfrak{H}(\mathfrak{Q}_n, ^*)$ and $\mathfrak{H}(\mathfrak{Q}'_m, ^{*\prime})$ to be isomorphic.

Theorem 5. *Suppose* $\mathfrak{J} = \mathfrak{H}(\mathfrak{Q}_n, ^*)$ *and* $\mathfrak{J}' = \mathfrak{H}(\mathfrak{Q}'_m, ^{*\prime})$ *are such that* $\mathfrak{Q}$ *and* $\mathfrak{Q}'$ *are split quaternion algebras and* $n, m \geqslant 3$. *Then the following three statements are equivalent*:

(i) *The lattices of inner ideals of* $\mathfrak{J}$ *and* $\mathfrak{J}'$ *are isomorphic.*

(ii) $\mathfrak{Q}_n$ *and* $\mathfrak{Q}'_m$ *are isomorphic rings.*

(iii) *There exists a semilinear Jordan algebra isomorphism of* $\mathfrak{J}$ *onto* $\mathfrak{J}'$.

Proof. Suppose the lattices of inner ideals are isomorphic; then by Lemma 9, $m = n$. As in the proof of Lemma 6, any isomorphism must map $\{e_1\mathfrak{J}e_1^*\}$ onto $\{e_2\mathfrak{J}'e_2^*\}$. Then it follows by Lemma 7 that there exists an inclusion-preserving bijection of $\{e_1\mathfrak{Q}_n \mid \text{rank } e_1 \geqslant 2\}$ onto $\{e_2\mathfrak{Q}'_n \mid \text{rank } e_2 \geqslant 2\}$. This is sufficient by the fundamental theorem of projective geometry for the existence of a ring isomorphism of $\mathfrak{Q}_n$ onto $\mathfrak{Q}'_n$. Thus (i) implies (ii).

The result that (ii) implies (iii) follows from the fact that $\mathfrak{J}$ and $\mathfrak{J}'$ are isomorphic if and only if $\mathfrak{Q}_n$ and $\mathfrak{Q}'_m$ are isomorphic [7, pp. 184-185].

It is clear that (iii) implies (i).

5. **The algebras** $\mathfrak{H}(\mathfrak{O}_3)$. In this section, we use the fundamental theorem of octonion planes [5, Chapter 3] to show that every automorphism of the lattice of inner ideals of an exceptional Jordan algebra $\mathfrak{H}(\mathfrak{O}_3)$ is induced by a norm semisimilarity.

We recall that the Jordan algebras $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$ can be obtained by a construction due to McCrimmon [13] from a cubic norm form $N(x)$, a quadratic mapping $x \to x^\#$ and a base point 1 satisfying certain axioms. An element x in $\mathfrak{J}$ is of *rank one* if $x \neq 0$ and $x^\# = 0$. It is clear that if x is of rank one then $\mathfrak{J}U_x = \Phi x$. The converse has recently been proved by McCrimmon [14]. Thus an element of rank one is a point as defined earlier.

McCrimmon [14] has proved that the inner ideals in $\mathfrak{H}(\mathfrak{O}_3)$ are either point spaces or spaces of the form $b \times \mathfrak{J}$ (where $x \times y = (x+y)^\# - x^\# - y^\#$) where b is an element of rank one. If $\mathfrak{O}$ is an octonion division algebra, then every point space is one dimensional. If $\mathfrak{O}$ is a split octonion algebra, then every maximal point space is of the form $\Phi e_1 + \mathfrak{O}\varepsilon[12]$ or $\Phi e_1 + \varepsilon\mathfrak{O}[12] + \Phi\varepsilon[13]$ in a suitable isotope where ε is a primitive idempotent in $\mathfrak{O}$ and $[ij]$ are as defined in the previous section.

We first quickly summarize some results due to Faulkner [5] concerning norm semisimilarities. Suppose $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$ and $\mathfrak{J}' = \mathfrak{H}(\mathfrak{O}_3')$ are defined over fields Φ and Φ' respectively. Let s be an isomorphism of Φ onto Φ'. If σ is a bijective s-semilinear mapping of $\mathfrak{J}$ onto $\mathfrak{J}'$ satisfying

$$(1) \qquad N(x\sigma) = \rho N(x)^s \quad \text{for all } x \text{ in } \mathfrak{J},$$

where $\rho \neq 0$ is fixed in Φ' and satisfying (1) for all field extensions Ω of Φ and Ω' of Φ' such that s can be extended to Ω, then σ is said to be an s-semisimilarity of $\mathfrak{J}$ onto $\mathfrak{J}'$.

We need not assume the result on field extension in the definition if Φ is an infinite field.

Faulkner has proved that σ is an s-semisimilarity of $\mathfrak{J}$ onto $\mathfrak{J}'$ if and only if there exists an s^{-1}-semilinear map σ' of $\mathfrak{J}'$ to $\mathfrak{J}$ satisfying

$$(2) \qquad \sigma' U_x \sigma = U'_{x\sigma} \quad \text{for all } x \text{ in } \mathfrak{J}.$$

If σ is a norm semisimilarity of $\mathfrak{J}$ to itself, we define $\hat{\sigma} = (\sigma')^{-1}$ where σ' is as in (2). Then it is well known that $\sigma \to \hat{\sigma}$ is an automorphism of order 2 in the group of norm semisimilarities of $\mathfrak{J}$.

We will require a few results on the lattice of inner ideals of $\mathfrak{H}(\mathfrak{O}_3)$ where $\mathfrak{O}$ is an octonion algebra.

Proposition 6. *Suppose $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$ where $\mathfrak{O}$ is an octonion algebra. A point space $\mathfrak{P}$ is contained in an inner ideal of the form $b \times \mathfrak{J}$ where b is of rank one if and only if $V_{b,a} = 0$ for all a in $\mathfrak{P}$.*

Proof. Faulkner [5] has proved that the group of norm-preserving transformations of $\mathfrak{J}$ acts transitively on $\{\Phi b \mid b \text{ of rank } 1\}$, that $(x \times y)\hat{\sigma} = x\sigma \times y\sigma$ and that $\sigma^{-1} V_{b,a} \sigma = V_{b\hat{\sigma},a\sigma}$ if σ is a norm-preserving transformation of $\mathfrak{J}$. Thus we can assume $b = e_1$. Then $b \times \mathfrak{J} = \mathfrak{J}_0(e_1)$, the Peirce zero-space of the primitive idempotent e_1. It is easily seen that $a \in \mathfrak{J}_0(e_1)$ if and only if $V_{e_1,a} = 0$.

McCrimmon [14] has proved that any point space of dimension at most 5 in $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$ where $\mathfrak{O}$ is a split octonion algebra can be imbedded in an inner ideal of the form $b \times \mathfrak{J}$. He has also shown that the inner ideals of the form $b \times \mathfrak{J}$ are maximal. We now settle the question of the maximality of the six-dimensional point spaces.

Proposition 7. *Suppose $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$ where $\mathfrak{O}$ is a split octonion algebra. Then the six-dimensional point spaces of $\mathfrak{J}$ are maximal inner ideals.*

Proof. Suppose $\mathfrak{P}$ is a point space contained in $b \times \mathfrak{J}$ where b is of rank one. Then there exists a norm-preserving transformation η of $\mathfrak{J}$ such that $(b \times \mathfrak{J})\eta = e_1 \times \mathfrak{J}$ [5, Lemma 3.1]. Then $\mathfrak{P}\eta \subseteq e_1 \times \mathfrak{J} = \mathfrak{J}_0(e_1)$. Then if $p \in \mathfrak{P}$, $p = \alpha e_2 + \beta e_3 + a[23]$ for some $\alpha, \beta \in \Phi$, $a \in \mathfrak{O}$. Then $p^\# = (\alpha\beta - n(a))e_1$. But $\alpha\beta - n(a)$ is a nondegenerate quadratic form on $\mathfrak{J}_0(e_1)$ and has maximal Witt index at most 5. Thus the dimension of $\mathfrak{P}$ is at most 5.

If $\mathfrak{O}$ is an octonion division algebra, the point spaces are the minimal inner ideals, and the inner ideals of the form $b \times \mathfrak{J}$ are the maximal inner ideals. Thus any automorphism of the lattice of inner ideals permutes $\{b \times \mathfrak{J} \mid b \text{ of rank } 1\}$ and permutes the point spaces. We would like to show that this last statement is true if $\mathfrak{O}$ is a split octonion algebra.

Suppose $\mathfrak{O}$ is a split octonion algebra. If $\mathfrak{C}$ is an inner ideal in $\mathfrak{H}(\mathfrak{O}_3)$, then we define $d(\mathfrak{C})$ to be the length of a maximal chain of inner ideals from $\mathfrak{C}$ to zero (excluding zero). Then if $\mathfrak{C}$ is a point space, $d(\mathfrak{C})$ is the usual dimension of $\mathfrak{C}$, and if $\mathfrak{C} = c \times \mathfrak{J}$, $d(\mathfrak{C}) = 6$ ([14] and Proposition 7).

Proposition 8. *Suppose φ is an automorphism of the lattice of inner ideals of $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$. Then φ permutes $\{b \times \mathfrak{J} \mid b$ of rank $1\}$, and φ permutes the set of point spaces.*

Proof. As we noted above, it suffices to assume $\mathfrak{O}$ is a split octonion algebra. If $\mathfrak{C}$ is an inner ideal in $\mathfrak{J}$, then $d(\mathfrak{C}\varphi) = d(\mathfrak{C})$. Thus it suffices to distinguish between the six-dimensional point spaces and the inner ideals of the form $c \times \mathfrak{J}$ in a lattice-theoretic way.

Suppose $\mathfrak{C}$ is an inner ideal of the form $c \times \mathfrak{J}$. Then by passing to an isotope, we can suppose $c = e_3$. Then $c \times \mathfrak{J} = \mathfrak{J}_0(e_3)$, the Peirce zero-space of the idempotent e_3. Let $\mathfrak{P}_1 = \Phi e_1 + \mathfrak{O}\varepsilon[12]$ and $\mathfrak{P}_2 = \Phi e_2 + \mathfrak{O}\bar{\varepsilon}[12]$ where ε is a primitive idempotent in $\mathfrak{O}$. Then it is clear that $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are point spaces contained in $\mathfrak{J}_0(e_3)$ and that $\mathfrak{P}_1 \cap \mathfrak{P}_2 = 0$. Thus any inner ideal of the form $c \times \mathfrak{J}$ contains two five-dimensional point spaces whose intersection is zero. This is clearly impossible inside a six-dimensional point space. Thus $\mathfrak{C}\varphi$ cannot be a point space, proving the proposition.

Now we begin to study the automorphism group of the lattice of inner ideals of $\mathfrak{H}(\mathfrak{O}_3)$.

Lemma 10. *Suppose η is an automorphism of the lattice of inner ideals of $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$. Then there exist bijections φ and ψ of the set $\{\Phi a \mid a$ of rank $1\}$ such that $(\Phi a)\eta = (\Phi a)\varphi$ and $(b \times \mathfrak{J})\eta = (\Phi b)\psi \times \mathfrak{J}$ where $V_{b,a} = 0$ if and only if $V_{(\Phi b)\psi,(\Phi a)\varphi} = 0$.*

Proof. Suppose η is an automorphism of the lattice of inner ideals. If $a \in \mathfrak{J}$ is of rank one, $(\Phi a)\eta = \Phi c$ for some c in $\mathfrak{J}$ of rank one. Then c is defined up to a scalar multiple, obtaining the mapping φ.

By Proposition 8, if $b \in \mathfrak{J}$ is of rank one, there exists d in $\mathfrak{J}$ of rank one such that $(b \times \mathfrak{J})\eta = d \times \mathfrak{J}$. Then d is defined up to a scalar multiple since $(b \times \mathfrak{J})^{\#} = \Phi b$ if b is of rank one [14, p. 467]. This clearly defines the mapping ψ.

Since η is invertible, it is clear that φ and ψ are bijections. The last statement of the lemma simply states $\Phi a \subseteq b \times \mathfrak{J}$ if and only if $(\Phi a)\varphi \subseteq (\Phi b)\psi \times \mathfrak{J}$ (Proposition 6).

We will need some results concerning octonion planes. A complete discussion of octonion planes is given in [5]. Suppose $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3)$ where $\mathfrak{D}$ is an octonion algebra. Let Π denote the set of elements of rank one in $\mathfrak{J}$. Consider two copies of $\Phi^* a$, $a \in \Pi$, $\Phi^* = \Phi - \{0\}$. Call one of them a_* (the *point* a), the other a^* (the *line* a). Faulkner defines various relations between points and lines. The most important one for us is x_* is *incident* to y^* if $V_{y,x} = 0$. This geometric structure is a plane called the *octonion plane*, denoted $\mathcal{P}(\mathfrak{J})$. A bijective mapping of the points of $\mathcal{P}(\mathfrak{J})$ to the points of $\mathcal{P}(\mathfrak{J}')$ (where $\mathfrak{J}' = \mathfrak{H}(\mathfrak{D}_3'))$ and the lines of $\mathcal{P}(\mathfrak{J})$ to the lines of $\mathcal{P}(\mathfrak{J}')$ is a *collineation* if it preserves the incidence relation.

Now suppose $\mathfrak{P}$ is a point space and b is an element of rank one; then $\mathfrak{P} \subseteq b \times \mathfrak{J}$ if and only if $V_{b,a} = 0$ for all a in $\mathfrak{P}$ (Proposition 6). Thus $\mathfrak{P} \subseteq b \times \mathfrak{J}$ if and only if a_* is incident to b^* for all a in $\mathfrak{P}$. This gives us a relation between the lattice of inner ideals of $\mathfrak{H}(\mathfrak{D}_3)$ and the geometry $\mathcal{P}(\mathfrak{J})$. As an immediate consequence of the above discussion, we obtain the following lemma.

Lemma 11. *Suppose we are given mappings*

$$\Phi a \to (\Phi a)\varphi, \qquad b \times \mathfrak{J} \to (\Phi b)\psi \times \mathfrak{J}$$

with $V_{b,a} = 0$ *if and only if* $V_{(\Phi b)\psi,(\Phi a)\varphi} = 0$ *as in Lemma* 10. *Then the mapping*

$$a_* \to (a\varphi)_*, \qquad b^* \to (b\psi)^*$$

is a collineation of $\mathcal{P}(\mathfrak{J})$.

Theorem 6. *Suppose* $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3)$ *where* $\mathfrak{D}$ *is an octonion algebra. Suppose* η *is an automorphism of the lattice of inner ideals of* $\mathfrak{J}$. *Then there exists a norm semisimilarity* σ *of* $\mathfrak{J}$ *such that* $\mathfrak{C}\eta = \mathfrak{C}\sigma$ *for every inner ideal* $\mathfrak{C}$ *of* $\mathfrak{J}$.

Proof. By Lemma 10, we know that given an automorphism η of the lattice of inner ideals there exist bijections φ and ψ of $\{\Phi a \mid a \text{ of rank } 1\}$ such that $(\Phi a)\eta = (\Phi a)\varphi$ and $(b \times \mathfrak{J})\eta = (\Phi b)\psi \times \mathfrak{J}$ with $V_{b,a} = 0$ if and only if $V_{(\Phi b)\psi,(\Phi a)\varphi} = 0$. Then by the fundamental theorem of octonion planes [5, Theorem 3.10], there exists a norm semisimilarity σ of $\mathfrak{J}$ such that $(a_*)\varphi = (a\sigma)_*$ and $(b^*)\psi = (b\hat{\sigma})^*$; but $(b \times \mathfrak{J})\sigma = b\hat{\sigma} \times \mathfrak{J}$ [5, p. 10], proving the theorem.

We now want to consider under what conditions the lattices of inner ideals of two algebras of the form $\mathfrak{H}(\mathfrak{D}_3)$ are isomorphic.

Proposition 9. *Suppose the lattice of inner ideals of $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3)$ is isomorphic to the lattice of inner ideals of $\mathfrak{J}' = \mathfrak{H}(\mathfrak{D}'_3)$. Then either $\mathfrak{D}$ and $\mathfrak{D}'$ are both split, or they are both division algebras. Moreover, there exists a norm semisimilarity of $\mathfrak{J}$ onto $\mathfrak{J}'$.*

Proof. Since the length of a maximal chain of inner ideals is different from $\mathfrak{H}(\mathfrak{D}_3)$ with $\mathfrak{D}$ split and $\mathfrak{D}$ a division algebra, the first statement is clear.

The second statement follows by noting that an isomorphism of the lattice of inner ideals of $\mathfrak{J}$ onto that of $\mathfrak{J}'$ is a collineation of $\mathcal{P}(\mathfrak{J})$ onto $\mathcal{P}(\mathfrak{J}')$ (the arguments are similar to those used in the proofs of Proposition 8 and Lemma 10), and then applying the fundamental theorem of octonion planes [5, Theorem 3.10].

Lemma 12. *The following two statements are equivalent*:

(i) *There exists a semilinear algebra isomorphism of $\mathfrak{D}$ onto $\mathfrak{D}'$.*

(ii) *There exists a norm semisimilarity of $\mathfrak{H}(\mathfrak{D}_3)$ onto $\mathfrak{H}(\mathfrak{D}'_3)$.*

Proof. The result that (i) implies (ii) follows from equations 1.33 of [5].

For the converse, suppose $\mathfrak{D}$ is a Φ-algebra, $\mathfrak{D}'$ is a Φ'-algebra, s is an isomorphism of Φ onto Φ' and σ is an s-semisimilarity of $\mathfrak{H}(\mathfrak{D}_3)$ to $\mathfrak{H}(\mathfrak{D}'_3)$. Define a Φ-algebra $\mathfrak{D}''$ to have the same set, addition and multiplication as $\mathfrak{D}'$ and Φ-action $\alpha x = \alpha^s x$ $(\alpha \in \Phi)$. It is clear that the identity map is an s^{-1}-semilinear isomorphism from $\mathfrak{D}$ to $\mathfrak{D}''$. Thus $\mathfrak{H}(\mathfrak{D}_3)$ and $\mathfrak{H}(\mathfrak{D}''_3)$ are norm similar. Then by the Albert-Jacobson theorem [5, Theorem 1.8] $\mathfrak{D}$ and $\mathfrak{D}''$ are isomorphic. Thus there exists a semilinear algebra isomorphism of $\mathfrak{D}$ onto $\mathfrak{D}'$.

Theorem 7. *Suppose $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3)$ and $\mathfrak{J}' = \mathfrak{H}(\mathfrak{D}'_3)$ are exceptional Jordan algebras. Then the lattice of inner ideals of $\mathfrak{J}$ is isomorphic to the lattice of inner ideals of $\mathfrak{J}'$ if and only if there exists a semilinear algebra isomorphism of $\mathfrak{D}$ onto $\mathfrak{D}'$.*

Proof. By Proposition 9, it is clear that the lattices of inner ideals of $\mathfrak{J}$ qnd $\mathfrak{J}'$ are isomorphic if and only if there exists a norm semisimilarity of $\mathfrak{J}$ onto $\mathfrak{J}'$. The result then follows from Lemma 12.

6. The algebras $\mathrm{Jord}(Q, 1)$ **.** In this section, we use some results due to Chow [2] and Dieudonné [3] to show that every automorphism of the lattice of inner ideals of a finite dimensional $\mathrm{Jord}(Q, 1)$ where Q is a nondegenerate quadratic form of Witt index at least three is induced by a semisimilarity of the form Q.

First, we quickly present some definitions concerning quadratic forms. Let Φ be a field and $\mathfrak{J}$ be a vector space over Φ with quadratic form Q. Suppose there exists an element 1 in $\mathfrak{J}$ such that $Q(1) = 1$. Then $(Q, 1)$ is said to be a *quadratic form with base point*. Define $Q(a,b) = Q(a+b) - Q(a) - Q(b)$, $T(a) = Q(a, 1)$, $\bar{a} = T(a)1 - a$. Then we define $xU_a = Q(a,\bar{x})a - Q(a)\bar{x}$. This makes $\mathfrak{J}$ into a quadratic Jordan algebra, denoted $\mathrm{Jord}(Q, 1)$ [9].

It is well known [9] that a in $\mathfrak{J}$ is invertible if and only if $Q(a) \neq 0$. If Q is a quadratic form on a vector space V, a subspace W is said to be *totally singular* if

$Q(w) = 0$ for all w in W. It is easily seen that a subspace $\mathfrak{C}$ of Jord$(Q, 1)$ is an inner ideal if and only if it is totally singular.

We say that a quadratic form Q on a vector space V is *regular* if 0 is the only element z in V such that $Q(a + z) = Q(a)$ for all a in V. A quadratic form Q is said to be *nondegenerate* if the radical of $Q(x,y)$ is equal to zero. If the characteristic of the base field is unequal to two, then Q is regular if and only if it is nondegenerate.

A semisimilarity is defined relative to quadratic forms in exactly the same way it was defined in the previous section for the cubic norm form $N(x)$. Jacobson and McCrimmon [9] have shown that the group of similarities of Jord$(Q, 1)$ is the structure group of that algebra.

We let $\mathcal{M}(V)$ denote the set of all maximal totally singular subspaces of V.

Lemma 13. *Let V be a finite dimensional vector space with regular quadratic form Q. Then every totally singular subspace of V is an intersection of maximal totally singular subspaces.*

Proof. Let U be a totally singular subspace of V with basis $u_1, \ldots, u_r$. Then it is well known that U can be imbedded in a maximal totally singular subspace $\mathfrak{M}$ of V. Let $u_1, \ldots, u_r, \ldots, u_s$ be a basis of $\mathfrak{M}$. Let $\mathfrak{W}$ be a subspace of V dual to $\mathfrak{M}$ with basis $w_1, \ldots, w_s$ consisting of singular vectors such that $Q(u_i, w_j) = \delta_{ij}$ and $Q(w_i, w_j) = 0$.

Then it is clear that $u_1, \ldots, u_r, w_{r+1}, \ldots, w_s$ is a basis of a totally singular subspace $\mathfrak{M}'$ which is maximal since it is of the same dimension as $\mathfrak{M}$.

Then it is clear that $U = \mathfrak{M} \cap \mathfrak{M}'$, proving the lemma.

Now suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two maximal totally singular subspaces of V. We define, following Chow [2], $\mathfrak{M}_1$ and $\mathfrak{M}_2$ to be *adjacent* if $\dim(\mathfrak{M}_1 \cap \mathfrak{M}_2) = \dim \mathfrak{M}_1 - 1$.

It is clear that if η is an automorphism of the lattice of inner ideals of $\mathfrak{J} = $ Jord$(Q, 1)$ then η is bijective adjacence-preserving transformation of $\mathcal{M}(\mathfrak{J})$.

The following proposition is due to Chow and Dieudonné [4, pp. 82-85].

Proposition 10. *Suppose V and V' are n-dimensional vector spaces with nondegenerate quadratic forms Q and Q' of equal Witt index at least three. Then every bijective transformation φ of $\mathcal{M}(V)$ onto $\mathcal{M}(V')$ such that φ and φ^{-1} are adjacence-preserving is induced by a semisimilarity of V onto V'.*

The assumption in Proposition 10 that the Witt index be at least three is indeed necessary as is explained in [4, p. 85].

Suppose φ is a bijective transformation of $\mathcal{M}(\mathfrak{J})$ such that φ and φ^{-1} are adjacence-preserving. Then it follows from Lemma 13 that φ can be extended in at most one way to an automorphism of the lattice of inner ideals of Jord$(Q, 1)$. However, it is clear that any semisimilarity of Jord$(Q, 1)$ induces an automorphism of the lattice of inner ideals. Thus we have the following theorem.

Theorem 8. *Let* $(Q, 1)$ *be a nondegenerate quadratic form of Witt index at least three on a finite-dimensional vector space* $\mathfrak{J}$. *Then every automorphism of the lattice of inner ideals of* $\mathrm{Jord}(Q, 1)$ *is induced by a semisimilarity.*

We now want to use Proposition 10 to deduce a necessary and sufficient condition for $\mathrm{Jord}(Q, 1)$ and $\mathrm{Jord}(Q', 1')$ of the same finite dimension to have isomorphic lattices of inner ideals.

Theorem 9. *Suppose* $\mathfrak{J} = \mathrm{Jord}(Q, 1)$ *and* $\mathfrak{J}' = \mathrm{Jord}(Q', 1')$ *are of the same finite dimension,* Q *and* Q' *are nondegenerate and one of them has Witt index at least three. Then the lattice of inner ideals of* $\mathrm{Jord}(Q, 1)$ *is isomorphic to the lattice of inner ideals of* $\mathrm{Jord}(Q', 1')$ *if and only if there exists a semisimilarity of* $\mathfrak{J}$ *onto* $\mathfrak{J}'$.

Proof. Suppose the lattices are isomorphic. Then, since the length of a maximal chain of inner ideals is the Witt index, the Witt indices of Q and Q' are equal. The result then follows from Proposition 10.

For the converse, it is clear that if σ is a semisimilarity of $\mathfrak{J}$ onto $\mathfrak{J}'$, then the mapping defined by $U\eta = \{u\sigma \mid u \in U\}$ is an isomorphism of the lattice of inner ideals of $\mathrm{Jord}(Q, 1)$ onto $\mathrm{Jord}(Q', 1')$.

7. **Nonisomorphism of the lattices of inner ideals.** In this section, we use many of the properties of the lattice of inner ideals to prove the following important theorem.

Theorem 10. *No two of the following types of simple quadratic Jordan algebras have isomorphic lattices of inner ideals*:

(1) *the algebras* Δ_n^+ *where* Δ *is a division ring and* $n \geqslant 3$;

(2) *the algebras* $\mathfrak{H}(\Delta_n, ^*)$ *where* Δ *is a division ring,* $n \geqslant 3$ *and* * *is a hermitian involution*;

(3) *the algebras* $\mathfrak{H}(\mathfrak{Q}_n, ^*)$ *where* $\mathfrak{Q}$ *is a split quaternion algebra and* $n \geqslant 3$;

(4) *the finite-dimensional algebras* $\mathrm{Jord}(Q, 1)$ *where* Q *is a regular quadratic form with base point* 1;

(5) *the algebras* $\mathfrak{H}(\mathfrak{O}_3)$ *where* $\mathfrak{O}$ *is an octonion division algebra*;

(6) *the algebras* $\mathfrak{H}(\mathfrak{O}_3)$ *where* $\mathfrak{O}$ *is a split octonion algebra.*

The proof of the theorem will be by a sequence of lemmas. By Proposition 9, the lattices of inner ideals of algebras of types (5) and (6) cannot be isomorphic.

A very important part of our argument will be the length of a maximal chain of inner ideals that begins with a maximal inner ideal. If

$$\mathfrak{C}_1 \supset \mathfrak{C}_2 \supset \ldots \supset \mathfrak{C}_r \supset \mathfrak{C}_{r+1} = 0$$

is such a chain, we say that the length of the chain is r. Then we have the following proposition whose proof is an easy consequence of the results we have developed on inner ideals.

Proposition 11. *The length of a maximal chain of inner ideals in the simple quadratic Jordan algebras is given by the following*:

(1) *in* Δ_n^+, *it is* $2n - 2$ $(n \geqslant 2)$;

(2) *in* $\mathfrak{H}(\Delta_n,^*)$, *it is* $n - 1$ $(n \geqslant 2)$;

(3) *in* $\mathfrak{H}(\mathfrak{Q}_n,^*)$, *it is* $2n - 1$ $(n \geqslant 3)$;

(4) *in* Jord$(Q, 1)$, *it is the Witt index of* Q;

(5) *in* $\mathfrak{H}(\mathfrak{O}_3)$ *where* $\mathfrak{O}$ *is an octonion division algebra, it is* 2;

(6) *in* $\mathfrak{H}(\mathfrak{O}_3)$ *where* $\mathfrak{O}$ *is a split octonion algebra, it is* 6.

Proof. Results (1) and (2) are respectively the second corollary on p. 458 and the second corollary on p. 462 of [14]. (3) is Lemma 9. (4) follows from the definition of Witt index. (5) and (6) follow from Theorem 8 of [14] and Proposition 7.

We are now ready to prove the sequence of lemmas which when put together will yield a proof of Theorem 10.

Lemma 14. *The lattice of inner ideals of* $\mathfrak{A}^+$ *where* $\mathfrak{A}$ *is simple artinian is not isomorphic to the lattice of inner ideals of* $\mathfrak{H}(\mathfrak{B},^*)$ *where* $\mathfrak{B}$ *is simple artinian and* * *is a hermitian involution.*

Proof. By Proposition 11, we see that $\mathfrak{B} = \Delta_n$ with $n \geqslant 3$. Suppose the lemma is false; then there exists an isomorphism η of the lattice of inner ideals of $\mathfrak{A}^+$ to the lattice of right ideals of $\mathfrak{B}$ (Proposition 4).

Suppose $e\mathfrak{A}$ and $\mathfrak{A}f$ are maximal inner ideals of $\mathfrak{A}^+$ with $(e\mathfrak{A})\eta = x\mathfrak{B}$ and $(\mathfrak{A}f)\eta = y\mathfrak{B}$. Then $(e\mathfrak{A}f)\eta = x\mathfrak{B} \cap y\mathfrak{B} = z\mathfrak{B}$ for some z in $\mathfrak{B}$.

But there exists (by an argument similar to that used to prove Proposition 2) a maximal right ideal $w\mathfrak{B} \supset z\mathfrak{B}$ with $w\mathfrak{B} \neq x\mathfrak{B}$ and $w\mathfrak{B} \cap y\mathfrak{B} = z\mathfrak{B}$. Then $(w\mathfrak{B})\eta^{-1}$ must be a maximal inner ideal in $\mathfrak{A}^+$, hence either a maximal left ideal or a maximal right ideal. It cannot be a left ideal since $e\mathfrak{A}f$ is not a left ideal. If it is a right ideal, then by Proposition 1, we must have $(w\mathfrak{B})\eta^{-1} = e\mathfrak{A}$ contradicting the fact that η is bijective.

As an immediate consequence of Proposition 11, we obtain the following lemma.

Lemma 15. *The lattice of inner ideals of* $\mathfrak{A}^+$ *where* $\mathfrak{A}$ *is simple artinian is not isomorphic to the lattice of inner ideals of* $\mathfrak{H}(\mathfrak{Q}_n,^*)$ *where* $\mathfrak{Q}$ *is a split quaternion algebra and* $n \geqslant 3$.

Lemma 16. *The lattice of inner ideals of* Δ_n^+ *where* Δ *is a division ring and* $n \geqslant 3$ *is not isomorphic to the lattice of inner ideals of a finite dimensional* $\mathfrak{J} =$ Jord$(Q, 1)$ *where* Q *is a regular quadratic form.*

Proof. Let $\mathfrak{M}_1$ be a maximal inner ideal in $\mathfrak{J}$. Then it is well known that there exists a maximal inner ideal $\mathfrak{M}_2$ such that $\mathfrak{M}_1 \cap \mathfrak{M}_2 = 0$.

Let φ be an isomorphism of the lattice of inner ideals of $\mathfrak{J}$ onto the lattice of inner ideals of Δ_n^+. Then $\mathfrak{M}_1\varphi$ and $\mathfrak{M}_2\varphi$ being maximal inner ideals in Δ_n^+ must be maximal one-sided ideals. However $\mathfrak{M}_1\varphi \cap \mathfrak{M}_2\varphi = 0$ with $\mathfrak{M}_1\varphi$ a right ideal and $\mathfrak{M}_2\varphi$ a left ideal (or vice versa) implies by Proposition 1 that either $\mathfrak{M}_1\varphi = 0$ or $\mathfrak{M}_2\varphi = 0$ which is clearly impossible since $\mathfrak{M}_1\varphi$ and $\mathfrak{M}_2\varphi$ are maximal. Thus either $\mathfrak{M}_1\varphi$ and $\mathfrak{M}_2\varphi$ are either both maximal left ideals or both maximal right ideals, but then their intersection cannot be zero since $n \geqslant 3$.

Lemma 17. *The lattice of inner ideals of* $\mathfrak{A}^+$, *a simple artinian, is not isomorphic to the lattice of inner ideals of* $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$ *where* $\mathfrak{O}$ *is an octonion algebra.*

Proof. There are two cases depending on whether $\mathfrak{O}$ is an octonion division algebra or a split octonion algebra.

Suppose first that $\mathfrak{O}$ is an octonion division algebra; then by Proposition 11 it is clear that the only $\mathfrak{A}$ for which an isomorphism can possible exist is $\mathfrak{A} = \Delta_2$. Thus every nonzero proper one-sided ideal in $\mathfrak{A}$ is mapped to an inner ideal of the form $b \times \mathfrak{J}$ where b is an element of rank one in $\mathfrak{J}$. Let e be a primitive idempotent in $\mathfrak{A}$. Then $e\mathfrak{A}$ and $(1-e)\mathfrak{A}$ are one-sided ideals in $\mathfrak{A}$, and their intersection is zero. But $(b \times \mathfrak{J}) \cap (c \times \mathfrak{J}) \neq 0$ for any b, c in $\mathfrak{J}$ of rank one since $b \times c \neq 0$ as the only point spaces are one dimensional.

Suppose $\mathfrak{O}$ is a split octonion algebra; then by Proposition 11 it follows that the only possible $\mathfrak{A}$ for which an isomorphism can exist is $\mathfrak{A} = \Delta_4$. Let η be an isomorphism of the lattice of inner ideals of $\mathfrak{J}$ onto the lattice of inner ideals of $\mathfrak{A}^+$. Then choose b and c in $\mathfrak{J}$ of rank 1 such that $b \times c \neq 0$. Then $(b \times \mathfrak{J}) \cap (c \times \mathfrak{J}) = \Phi(b \times c)$, a minimal inner ideal. Then $(b \times \mathfrak{J})\eta$ and $(c \times \mathfrak{J})\eta$ must be maximal one-sided ideals in Δ_4 whose intersection is a minimal inner ideal in Δ_4^+, a contradiction of Proposition 1 since a minimal inner ideal in Δ_4^+ is an intersection of a minimal left ideal and a minimal right ideal.

Lemma 18. *The lattice of inner ideals of* $\mathfrak{J}_1 = \mathfrak{H}(\Delta_m, {}^*)$ *where* Δ *is a division ring and* * *is a hermitian involution is not isomorphic to the lattice of inner ideals of* $\mathfrak{J}_2 = \mathfrak{H}(\mathfrak{Q}_n, {}^{*'})$ *where* $\mathfrak{Q}$ *is a split quaternion algebra and* $n \geqslant 3$.

Proof. By Proposition 11, $m \geqslant 6$. By Proposition 4, the lattice of inner ideals of $\mathfrak{J}_1$ is isomorphic to the lattice of right ideals of Δ_m. Thus it suffices to show that the lattice of right ideals of Δ_m is not isomorphic to the lattice of inner ideals of $\mathfrak{J}_2$. In $\mathfrak{J}_2$, there exist two maximal inner ideals, namely $\Phi e_1 + \varepsilon\mathfrak{Q}[12] + \ldots + \varepsilon\mathfrak{Q}[1n]$ and $\Phi e_1 + \bar{\varepsilon}\mathfrak{Q}[12] + \ldots + \bar{\varepsilon}\mathfrak{Q}[1n]$ (where ε is a primitive idempotent in $\mathfrak{Q}$), intersecting in a minimal inner ideal, but that cannot happen in Δ_m since $m \geqslant 6$.

Lemma 19. *The lattice of inner ideals of* $\mathfrak{J}_1 = \mathfrak{H}(\Delta_n, {}^*)$ *where* $n \geqslant 3$ *and* * *is a hermitian involution is not isomorphic to the lattice of inner ideals of a finite dimensional* $\mathfrak{J}_2 = \mathrm{Jord}(Q, 1)$.

Proof. It is known that the lattice of inner ideals of $\mathfrak{J}_1$ is isomorphic to the lattice of right ideals of Δ_n. As in the proof of Lemma 16, it is clear that there cannot be two maximal right ideals in Δ_n with zero intersection since $n \geqslant 3$. Thus we cannot have an isomorphism in this case.

Lemma 20. *The lattice of inner ideals of $\mathfrak{H}(\Delta_n, {}^*)$ where Δ is a division ring and * is a hermitian involution is not isomorphic to the lattice of inner ideals of $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3)$ where $\mathfrak{D}$ is an octonion division algebra.*

Proof. By Proposition 11, it follows that the only possible n for which an isomorphism can occur is $n = 3$.

The lattice of inner ideals of $\mathfrak{H}(\Delta_3, {}^*)$ is isomorphic to the lattice of right ideals of Δ_3 (Proposition 4). The lattice of right ideals of Δ_3 is isomorphic to the lattice of left ideals of Δ_3^{op} which is isomorphic to the lattice of subspaces of a three-dimensional vector space V over Δ^{op}. On this last lattice, one constructs in the usual way the structure of a projective plane. It is well known [6, Chapter 20] that this projective plane is desarguesian since Δ^{op} is associative.

As a consequence of Proposition 6 and the discussion of octonion planes given in §5, it follows that if $\mathfrak{D}$ is an octonion division algebra, $\mathscr{P}(\mathfrak{J})$ is simply a projective plane constructed on the lattice of inner ideals of $\mathfrak{J}$. This plane is known to be nondesarguesian [5, p. 44].

If the lattice of inner ideals of $\mathfrak{J}$ were isomorphic to the lattice of right ideals of Δ_3, the corresponding projective planes would be isomorphic since the geometric relations are determined by inclusion. Thus no such isomorphism can exist.

Lemma 21. *The lattice of inner ideals of $\mathfrak{H}(\Delta_n, {}^*)$ where Δ is a division ring and * is a hermitian involution is not isomorphic to the lattice of inner ideals of $\mathfrak{H}(\mathfrak{D}_3)$ where $\mathfrak{D}$ is a split octonion algebra.*

Proof. Suppose the lemma is false; then by Proposition 11, $n = 7$. Let η be an isomorphism of the lattice of inner ideals of $\mathfrak{H}(\mathfrak{D}_3)$ onto the lattice of right ideals of Δ_7 (Proposition 4). Then b, c in $\mathfrak{H}(\mathfrak{D}_3)$ are of rank one with $b \times c \neq 0$. Then $(\Phi b)\eta$ and $(\Phi c)\eta$ are minimal right ideals, say $e_1\mathfrak{A}$ and $e_2\mathfrak{A}$ respectively (where $\mathfrak{A} = \Delta_7$). Then it is clear that $e_1\mathfrak{A} + e_2\mathfrak{A}$ cannot be a maximal right ideal. However, the smallest inner ideal containing Φb and Φc is a maximal inner ideal, a contradiction.

Lemma 22. *The lattice of inner ideals of $\mathfrak{J}_1 = \mathfrak{H}(\mathfrak{Q}_n, {}^*)$ where $\mathfrak{Q}$ is a split quaternion algebra and $n \geqslant 3$ is not isomorphic to the lattice of inner ideals of a finite-dimensional $\mathfrak{J}_2 = \mathrm{Jord}(Q, 1)$ where Q is a regular quadratic form.*

Proof. The $(2n - 1)$-dimensional point spaces and the inner ideals $e\mathfrak{J}_1e^*$ where e is an idempotent of rank $2n - 1$ in $\mathfrak{Q}_n$ are maximal inner ideals which are not conjugate to each other under the automorphism group of the lattice of inner ideals of $\mathfrak{J}_1$ (Lemma 6). By Witt's theorem, any two maximal inner ideals of $\mathfrak{J}_2$ are conjugate under the automorphism group of the lattice of inner ideals of $\mathfrak{J}_2$. Hence the lattices of inner ideals of $\mathfrak{J}_1$ and $\mathfrak{J}_2$ cannot be isomorphic.

As an immediate consequence of Proposition 11, we obtain the following lemma.

Lemma 23. *The lattice of inner ideals of* $\mathfrak{H}(\mathfrak{Q}_n,{}^*)$ *where* $\mathfrak{Q}$ *is a split quaterion algebra and* $n \geq 3$ *is not isomorphic to the lattice of inner ideals of* $\mathfrak{H}(\mathfrak{O}_3)$ *where* $\mathfrak{O}$ *is an octonion algebra.*

Lemma 24. *The lattice of inner ideals of a finite-dimensional* $\mathfrak{J}_1 = \mathrm{Jord}(Q, 1)$ *where* Q *is a regular quadratic form is not isomorphic to the lattice of inner ideals of* $\mathfrak{J}_2 = \mathfrak{H}(\mathfrak{O}_3)$ *where* $\mathfrak{O}$ *is an octonion algebra.*

Proof. The inner ideals of the form $b \times \mathfrak{J}_2$ where b is an element of rank one are maximal inner ideals in $\mathfrak{J}_2$. By a result on the geometry of octonion planes [5, Lemma 3.4], any two such inner ideals have a nonzero intersection. However it is known that there exist maximal inner ideals $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in $\mathfrak{J}_1$ such that $\mathfrak{M}_1 \cap \mathfrak{M}_2 = 0$. Thus no such isomorphism can exist.

The proof of Theorem 11 is an immediate consequence of Lemmas 14-24 and Proposition 9.

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